

Functorial quantum field evolution in discretised spacetime



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2. In particular, we aim at fixing the minimum number of properties we expect a QFT to possess in a discrete spacetime and examine what additional structure we can extract from them.
3. The basic notions we rely upon is the causal structure of spacetime¹ and the way quantum systems compose².

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3. A familiar example is the category **Hilb**, where objects are Hilbert spaces and morphisms are linear maps.

4. A category also includes

- ▶ a means of sequential composition of morphisms, i.e. for $f : A \rightarrow B$ and $g : B \rightarrow C$ there is a morphism $h = g \circ f$.
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8. In physics we think of $f \otimes g$ as a joint process running in parallel. The tensor product encodes a notion of space.
9. There is a tensor unit such that $I \otimes A \cong A$ and $B \otimes I \cong B$

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between two categories \mathbf{C} and \mathbf{D} such that

- ▶ Objects X in \mathbf{C} are associated to objects $F(X)$ in \mathbf{D} .
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12. Preservation of sequential composition is related to time. Monoidality is related to space.

From a functorial perspective a QFT is a map

causal structure \rightarrow algebra

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2. When a is not related to b , we say that a and b are spacelike separated.
3. A *causal path* from x to y is a totally ordered subset $\gamma \subseteq C$ such that $x = \min \gamma$ and $y = \max \gamma$.

4. We define a *slice* Σ in (C, \leq) to be an antichain, i.e. a subset $\Sigma \subseteq C$ such that any two distinct $x, y \in \Sigma$ are space-like separated.

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5. A *Cauchy slice*, is a maximal slice, i.e. a slice Σ such that every $x \in C$ is causally related to some $y \in \Sigma$.
6. The *future domain of dependence* $D^+(\Sigma)$ of Σ is $D^+(\Sigma) := \{x \in C \mid \text{such that every past-directed causal path through } x \text{ intersects } \Sigma\}$.

Let $\Omega := (C, \leq)$ be a causal set. The *category of all slices* on Ω , denoted by **Slice** $[\Omega]$, is the monoidal category defined as follows:

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4. The identity morphism is $\Sigma \preceq \Sigma$.

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2. The monoidal unit for the tensor product is the empty slice $\emptyset \subseteq C$.
3. The partial monoidal product on objects immediately extends to morphisms because:

$$\Sigma' \subseteq D^+(\Sigma) \text{ and } \Gamma' \subseteq D^+(\Gamma) \quad \Rightarrow$$

$$\Sigma' \sqcup \Gamma' \subseteq D^+(\Sigma) \cup D^+(\Gamma) \subseteq D^+(\Sigma \sqcup \Gamma)$$

A *causal field theory* Ψ on Ω is a monoidal functor

$$\Psi: \mathbf{Slice}[\Omega] \rightarrow \mathbf{Hilb}.$$

1. On objects, Ψ associates each slice Σ to the Hilbert space $\Psi(\Sigma) := H_\Sigma$ of fields over that slice.

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What about a more quantum informational perspective?
Category of C^* -algebras and CPTP maps.

5. The empty set of $\text{Slice}[\Omega]$ is mapped to the field of complex numbers.

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6. Monoidality of the functor on objects says that Ψ associates the disjoint union of slices $\Sigma \sqcup \Gamma$ to the tensor product $H_\Sigma \otimes H_\Gamma$ of the spaces of fields on the individual slices.

Proposition: Let $\Sigma \preceq \Sigma'$ and $\Gamma \preceq \Gamma'$ for space-like separated slices Σ and Γ and space-like separated slices Σ' and Γ' . Consider the field evolution between the disjoint unions of slices:

$$\Psi((\Sigma \otimes \Gamma) \rightarrow (\Sigma' \otimes \Gamma')) : \Psi(\Sigma) \otimes \Psi(\Gamma) \rightarrow \Psi(\Sigma') \otimes \Psi(\Gamma')$$

Monoidality on morphisms implies that the field evolution above factors as the product of the individual field evolutions $\Psi(\Sigma) \rightarrow \Psi(\Sigma')$ and $\Psi(\Gamma) \rightarrow \Psi(\Gamma')$:

$$\Psi((\Sigma \otimes \Gamma) \rightarrow (\Sigma' \otimes \Gamma')) = \Psi(\Sigma \rightarrow \Sigma') \otimes \Psi(\Gamma \rightarrow \Gamma')$$

1. We define the following family of effects, indexed by all slices $\Sigma \in \mathcal{C} \dashv_{\Sigma}$

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1. We define the following family of effects, indexed by all slices $\Sigma \in \mathcal{C}$ $\dashv\Sigma$

$$\dashv\Sigma := \Psi(\Sigma \rightarrow \emptyset)$$

2. By monoidality we can prove that

$$\dashv\Sigma \otimes \Gamma = \dashv\Sigma \otimes \dashv\Gamma$$

3. The effects $\dashv\Sigma$ have a clear operational meaning. They remove the system from our scope:

$$\Psi(\Sigma \rightarrow \Gamma) \circ \dashv\Gamma = \Psi(\Sigma \rightarrow \Gamma) \circ \Psi(\Gamma \rightarrow \emptyset) = \Psi(\Sigma \rightarrow \emptyset) = \dashv\Sigma$$

The identification of functorial action with field evolution is the core idea of this work.

Causal structure	QFT
Slices (space)	Hilbert spaces (states)
Causal relationships (spacetime)	Linear operators (processes)
Slice composition	Process composition
Identity on Slices	Identity process

We accomodate general kind of processes (A possible venue for measurement theory in QFT?)

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We are currently building connections with QCAs and AQFT.

Thank you!